TECHNICAL nEPORT 3413

ESTIMATING THE MEAN
AND
STANDARD DEVIATION
FROM A
BRUCETON STATISTICAL ANALYSIS

LEONARD SHAINHEIT

AUGUST 1966

PICATINNY ARSENAL
DOVER, NEW JERSEY

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ACKNOWLEDGMENT

SUMMARY

The maximum likelihood equations used to determine estimates of the mean and standard deviation for the Bruceton method of statistical analysis were solved numerically. A digital computer algorithm was developed to reduce the time associated with the numerical calculations. The method of successive approximations (Newton-Raphson method) was suitable. Test data was evaluated to determine the relative difference between values of the means and standard deviation as obtained from the numerical solution of the maximum likelihood equations and as obtained from formulae approximating these equations. Results of the analysis showed that the differences between means were slight while those between standard deviations were more pronounced. Calculated high and low probability levels, because of their dependence on the mean and standard deviation, differed widely from one method of calculation to the other. The analysis also showed that the most adverse effects occurred for the smallest samples. Because the maximum likelihood equations yield better estimates of the mean and standard deviation than can be obtained from equations approximating them -- especially for small samples -- solution of the maximum likelihood equations by the Newton-Raphson method is advocated for determining the best estimates of the population mean and standard deviation.

CONCLUSIONS

The maximum likelihood equations used to estimate the mean and standard deviation in conjunction with a Bruceton analysis are readily solvable by the Newton-Raphson method of successive approximations with the aid of the digital computer codes of Appendix A and B.

The code of Appendix B appears to be superior based on higher reliability and safety criteria.

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RECOMMENDATION

Because of its apparent superiority, the code of Appendix B should be used to calculate the maximum likelihood estimates of the mean and standard deviation for samples analyzed by the Bruceton method.

BACKGROUND

The Bruceton method of statistical analysis is used to determine the probability of explosion or non-explosion for a given explosive item as a function of input stimuli. The procedures used for calculating the statistical parameters (mean, standard deviation, percent points, etc.) are based on approximations to the maximum likelihood equations which will be discussed at greater length in the Study section. However, the validity of the approximations fall off rapidly with decrease in sample sizes less than 100. Consequently, it is desirable to use the maximum likelihood equations for small sample analysis.

Before the advent of the digital computer, exact solutions of the maximum likelihood equations required complex, wholly impractical computational schemes. As a result of the difficulties formerly encountered in applying the likelihood theory, approximate methods were always used to estimate the maximum likelihood quantities. In the case of small sample analysis, blatant deviations frequently occurred between sample and population statistical parameters. The Franklin Institute of Philadelphia, Pennsylvania investigated the affect of approximate methods on the data obtained from small samples and their findings indicate that the expected deterioration of the approximate formulae as applied to small samples, does in fact occur.

It must be emphasized that in any statistical analysis large samples are preferable. But if small samples must be used in a Bruceton test, the use of maximum likelihood techniques become in a realistic sense, mandatory. Furthermore, the maximum likelihood values associated with small samples should be treated with care, since small samples tend to yield poor statistical results. This is especially true for this particular problem.

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STUDY

A method of statistical analysis frequently utilized in evaluating explosive devices is known as the Bruceton Staircase Sensitivity Test. The method was developed by the Explosives Research Laboratory of Bruceton, Pennsylvania. It is used to ascertain the safe and sure-fire energy levels associated with a given explosive item. In performing the experimental procedure the tester is first required to home in on a 50% firing level (mean) and on an approximate standard deviation (interval between consecutive energy levels). This can be accomplished easily if the past history of the explosive under investigation is available. Otherwise trial and error techniques are adequate. Once having chosen the 50% firing level and the step interval (based on an approximation of the standard deviation) the tester may then proceed as follows:

An attempt is made to actuate the explosive using the 50% energy estimate. If successful a second specimen is tested at an energy level one step interval below the 50% value. On the other hand, if the specimen failed to detonate, the second explosive item is tested at an energy level one step interval above the 50% level. Thus denoting any test level as H_i and the step interval as d, then $H_{i+1} = H_i - d$, for the case where the i' th test produced an explosion and $H_{i+1} = H_i + d$ if the i' th test produced a non-explosion.

In this way, a sequence of explosions (xs) and non-explosions (0s) will be generated. Figure 1 shows a typical sequence. It may be noted that the bulk of the testing occurs in the neighborhood of the mean energy level. Thus a minimal number of tests will permit a good estimation of the mean. However, because of the approximate nature of the methods of calculation used to estimate the mean and standard deviation, samples of at least 100 specimens are required. Also, sensitivity data does not consist of exact measurements. Consequently the information contained in them is reduced by about one-half that which would be contained in measured data of a comparable nature.
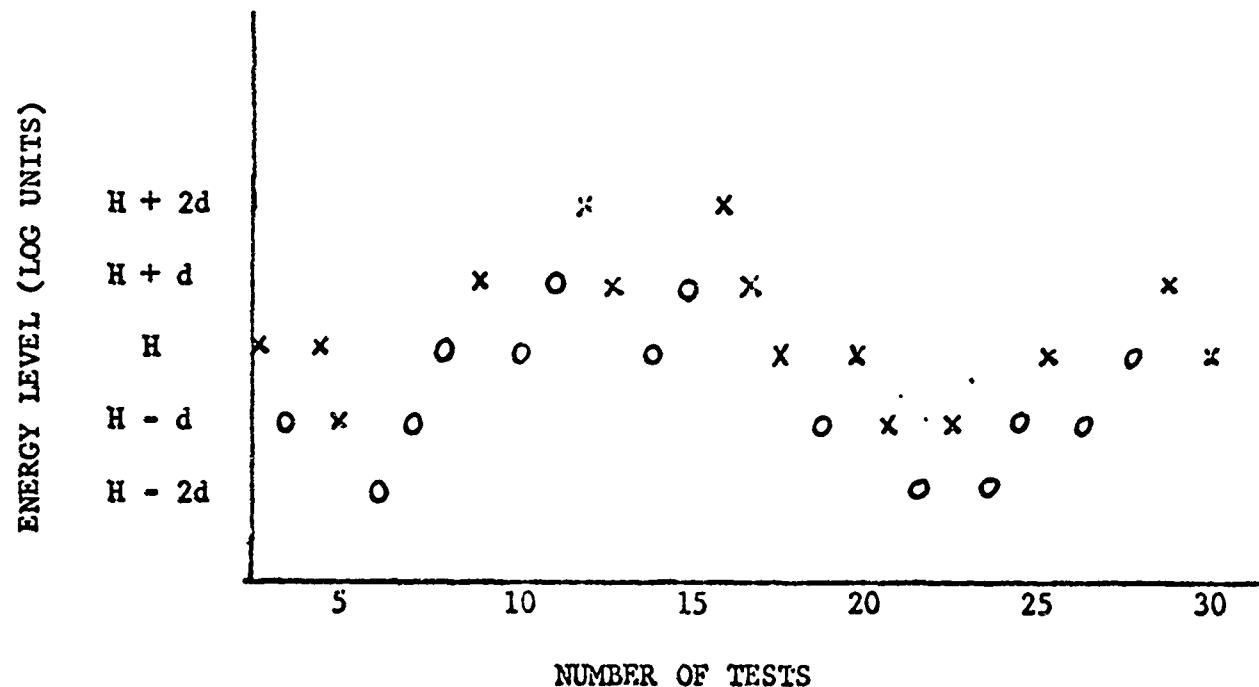
TYPICAL SEQUENCE OF EXPLOSIONS (X) AND NONEXPLOSIONS (O)
GENERATED IN A BRUCETON ANALYSIS

FIGURE 1

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To reduce the bias associated with the Bruceton method, large sample sizes (at least 100 specimens) should be used. For very expensive explosive devices, large sample sizes are prohibitive. Consequently, tests are conducted using smaller sample sizes often 50 specimens or less. But it is known that for sample sizes below 50, the approximations of the mean and standard deviation become rather poor. Therefore, more elaborate methods using the maximum likelihood equations for estimating the mean and standard deviation must be used. However, these equations cannot be solved in closed form by any direct method (Equation 47, p. 38 of Reference 4). Of course, numerically the equations can be solved and the maximum likelihood estimates of the mean and standard deviation determined.

Owing to the time involved in the numerical solution of the maximum likelihood equations, no attempt was made to utilize them until the advent of the digital computer. However, once the computer algorithm developed in this report was established it became a rather simple task to demonstrate the effect of sample size on the calculated mean and standard deviation especially since it is this parameter which strongly affects the outcome of the extrapolation calculations. That is, the validity of the approximation formulae for estimating high and low percent points is doubtful -- especially if small sample sizes are used. Therefore, the purpose of this report is to present a numerical procedure for solving the maximum likelihood equations and suitable for application to a digital computer. The effect of sample size on the results of both the approximate and the numerical techniques will be emphasized.

To determine the maximum likelihood equations for estimating the mean and standard deviation of a normally distributed random variable x with mean A and variance S^2, a random sample of size N is drawn from the population and tested in the manner described so that a sequence of ones and zeros is obtained. If:

N_1 = total number of ones

N_0 = total number of zeros

H_i = energy at test level i

N_i = number of ones at H_i

M_i = number of zeros at H_i

then

$$N_i = \sum N_i$$
$$N_0 = \sum M_j$$

and $\quad N = N_0 + N_1$

Further, it may be assumed $N_1 < N_0$, then the probability that an X will be a one at the H_i level is:

$$P_i = \frac{1}{\sqrt{2\pi}S} \int_{-\infty}^{H_i} e^{-\frac{(x-A)^2}{2S^2}} dx$$

and the probability that X will be a zero is: $Q_i = 1 - P_i$.

Then, the probability of generating a sequence of N_1 ones and N_0 zeros is obtained from:

$$P = K \prod_i P_i^{N_i} Q_i^{M_i}$$

where K is a function of the N_i and M_i. This probability is based on generating a sequence of N independent outcomes (ones and zeros) from a multi-nomial distribution. Although this assumption is not strictly applicable to the current problem, only slight adjustments need be made to the general theory to apply it in the asymptotic case. This is denoted by the likelihood:

$$L = \sum N_i \, LOG \, P_i Q_{i-1}$$

and differentiating with respect to A and S maximized likelihood equations are obtained. Before proceeding further the following derivatives are defined:

$$\frac{\partial P_i}{\partial A} = \frac{1}{S} Z_i$$

$$\frac{\partial P_i}{\partial S} = -\frac{H_i - A}{S^2} Z_i$$

$$\frac{\partial Q_i}{\partial A} = \frac{1}{S} Z_i$$

and $$\frac{\partial Q_i}{\partial S} = \frac{H_i - A}{S^2} Z_i$$

where $$Z_i = \frac{1}{\sqrt{2\pi} S} e^{-\frac{(H_i - A)^2}{2 S^2}}$$

Then, the maximum likelihood equations for estimating the mean and standard deviation are:

$$\frac{\partial L}{\partial A} = \frac{1}{S} \sum_i N_i \left[\frac{Z_{i-1}}{Q_{i-1}} - \frac{Z_i}{P_i} \right] = 0 \tag{1}$$

and

$$\frac{\partial L}{\partial S} = \frac{1}{S^2} \sum_i N_i \left[\frac{(H_{i-1} - A)Z_{i-1}}{Q_{i-1}} - \frac{(H_i - A)Z_i}{P_i} \right] = 0 \tag{2}$$

These equations cannot be solved by direct methods. However, if good estimates of the mean and standard deviation are available a method of successive approximations (Newton-Raphson method) is applicable. The method is as follows: Let the maximum likelihood Equation (1) and (2) be:

$$F_1 (S, A) = 0 \tag{3}$$

$$F_2 (S, A) = 0 \tag{4}$$

Also, let A_0, S_0 be approximate solutions of Equation (3) and (4) such that the true roots of Equation (3) and (4), A and S are related to A_0 and S_0 by:

$$S = S_0 + H$$
$$A = A_0 + K$$

where H and K are the corrections for S_0 and A_0, respectively. Rewriting Equation (3) and (4):

$$F_1 (S_0 + H, A_0 + K) = 0 \tag{5}$$

$$F_2 (S_0 + H, A_0 + K) = 0 \tag{6}$$

Equation (5) and (6) may be expanded by Taylor's theorem for a function of two variables giving:

$$F_1 (S_0 + H, A_0 + K) = 0 = F_1 (S_0, A_0) + H\left(\frac{\partial F_1}{\partial S}\right)_0 + K\left(\frac{\partial F_1}{\partial A}\right)_0 +$$

Terms in Higher Powers of H, K

$$F_2 (S_0 + H, A_0 + K) = 0 = F_2 (S_0, A_0) + H\left(\frac{\partial F_2}{\partial S}\right)_0 + K\left(\frac{\partial F_2}{\partial A}\right)_0 +$$

Terms in Higher Powers of H, K

Since the values of H and K are small, their squares, products, and higher powers are neglected so that:

$$F_1 (S_0, A_0) + H\left(\frac{\partial F_1}{\partial S}\right)_0 + K\left(\frac{\partial F_2}{\partial A}\right)_0 = 0 \tag{7}$$

$$F_2 (S_0, A_0) + H\left(\frac{\partial F_1}{\partial S}\right)_0 + K\left(\frac{\partial F_2}{\partial A}\right)_0 = 0 \tag{8}$$

Using Cramer's rule (assuming $D \neq 0$) solution for H and K are obtained from:

$$H_0 = \frac{DET \begin{bmatrix} -F_1(S_0, A_0) & \left(\frac{\partial F_1}{\partial A}\right)_0 \\ -F_2(S_0, A_0) & \left(\frac{\partial F_2}{\partial A}\right)_0 \end{bmatrix}}{D} \tag{9}$$

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and

$$K_o = \cfrac{DET \begin{bmatrix} \left(\dfrac{\partial F_1}{\partial S}\right)_o & -F_1(S_o, A_o) \\[2ex] \left(\dfrac{\partial F_2}{\partial S}\right)_o & -F_2(S_o, A_o) \end{bmatrix}}{D} \qquad (10)$$

where

$$D = \begin{bmatrix} \left(\dfrac{\partial F_1}{\partial S}\right)_o & \left(\dfrac{\partial F_1}{\partial A}\right)_o \\[2ex] \left(\dfrac{\partial F_2}{\partial S}\right)_o & \left(\dfrac{\partial F_2}{\partial A}\right)_o \end{bmatrix}$$

Using the values of H and K obtained above new approximations are gotten for S and A as:

$$S_1 = S_0 + H_0$$

$$A_1 = A_0 + K_0$$

If the process is repeated using S_1 and A_1, new corrections, H_1 and K_1, are obtained. Continuing in this manner, a predetermined degree of convergence may be achieved. The subscripts $0, 1, \ldots, n$, refer to the values A, S, K, and H to be used. That is $\left(\dfrac{\partial F_1}{\partial S}\right)_0$ means that $\dfrac{\partial F_1}{\partial S}$ is to be evaluated for $S = S_0$, $A = A_0$, and so on.

Also, examination of Equation (7), (8), (9) and (10) shows that for large D and correspondingly small H and K, the assumptions made are applicable and convergence should be expected.

In accordance with this method, Equation (1) and (2) must be differentiated partially with respect to S and A. However, before proceeding the following quantities are needed:

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$$\frac{\partial z_i}{\partial S} = \left[\frac{(H_i-A)^2 - S^2}{S^3}\right] z_i$$

$$\frac{\partial z_i}{\partial A} = \frac{(H_i-A)}{S^2}\, z_i$$

$$\frac{\partial\left(\frac{1}{Q_i}\right)}{\partial A} = -\frac{z_i}{Q_i^2 S}$$

$$\frac{\partial\left(\frac{1}{Q_i}\right)}{\partial S} = -\frac{(H_i-A)}{(Q_i S)^2}\, z_i$$

$$\frac{\partial\left(\frac{1}{P_i}\right)}{\partial A} = \frac{z_i}{P_i^2 S}$$

$$\frac{\partial\left(\frac{1}{P_i}\right)}{\partial S} = \frac{(H_i-A)}{(P_i S)^2}\, z_i$$

Then for Equation (1):

$$\frac{\partial F_1}{\partial S} = \frac{1}{S}\sum_1 N_i\left[\frac{\partial z_{i-1}}{\partial S}\left(\frac{1}{Q_{i-1}}\right) + \frac{\partial\left(\frac{1}{Q_{i-1}}\right)}{\partial S} z_{i-1} - \frac{\partial z_i}{\partial S}\left(\frac{1}{P_i}\right) - \frac{\partial\left(\frac{1}{P_i}\right)}{\partial S} z_i\right] - \frac{1}{S^2}\sum_1 N_i\left[\frac{z_{i-1}}{Q_{i-1}} - \frac{z_i}{P_i}\right],$$

and

$$\frac{\partial F_1}{\partial A} = \frac{1}{S}\sum_i N_i\left[\frac{\partial z_{i-1}}{\partial A}\left(\frac{1}{Q_{i-1}}\right) + \frac{\partial\left(\frac{1}{Q_{i-1}}\right)}{\partial A} z_{i-1} - \frac{\partial z_i}{\partial A}\left(\frac{1}{P_i}\right) + \frac{\partial\left(\frac{1}{P_i}\right)}{\partial A} z_i\right].$$

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Repeating the preceding for Equation (2) the following results are obtained:

$$\frac{\partial F_z}{\partial S} = \frac{1}{S^2}\sum_i N_i \left[\frac{(H_{i-1}-A)}{Q_{i-1}}\frac{\partial Z_{i-1}}{\partial S} + (H_{i-1}-A)\frac{\partial\left(\frac{1}{Q_{i-1}}\right)}{\partial S}Z_{i-1} - \right.$$

$$\left. \frac{(H_i - A)}{P_i}\frac{\partial Z_i}{\partial S} - (H_i - A)\frac{\partial\left(\frac{1}{P_i}\right)}{\partial S}Z_i \right] -$$

$$\frac{2}{S^3}\sum_i N_i\left[\frac{(H_{i-1}-A)}{Q_{i-1}}Z_{i-1} - \frac{(H_i-A)}{P_i}Z_i \right],$$

and

$$\frac{\partial F_z}{\partial A} = \frac{1}{S^2}\sum_i N_i \left[\frac{H_{i-1}}{Q_{i-1}}\frac{\partial Z_{i-1}}{\partial A} + H_{i-1}\frac{\partial\left(\frac{1}{Q_{i-1}}\right)}{\partial A}Z_{i-1} \right.$$

$$- \frac{Z_{i-1}}{Q_{i-1}} - \frac{A}{Q_{i-1}}\frac{\partial Z_{i-1}}{\partial A} - A Z_{i-1}\frac{\partial\left(\frac{1}{Q_{i-1}}\right)}{\partial A}$$

$$- \frac{H_i}{P_i}\frac{\partial Z_i}{\partial A} - \frac{\partial\left(\frac{1}{P_i}\right)}{\partial A}H_i Z_i + \frac{Z_i}{P_i} + A Z_i$$

$$\left. \frac{\partial\left(\frac{1}{P_i}\right)}{\partial A} - \frac{A}{P_i}\frac{\partial Z_i}{\partial A} \right].$$

Using Equation (9) and (10) repeatedly, corrections (H and K) are calculated for S and A until the desired degree of convergence is achieved. Of course, owing to the lengthy calculations required by this method, a digital computer program for working out the details was essential. Otherwise, the method would prove impractical.

Two similar FORTRAN codes for calculating the results of a
Bruceton analysis are in Appendix A and B. The two programs
are basically the same. A comparison of them follows.

The main program entitled BRUCE in both codes is used to
read in information pertaining to reliability and confidence levels.
Then subroutine FOREST is called. (This subroutine is a
modification of a FORTRAN code written by Forrest L. McMains,
(Reference 3).)

Subroutine FOREST is used to read in the basic data generated
in the Bruceton analysis and then calculate estimates of the mean
and standard deviation from approximations of the maximum likeli-
hood equations. However, the estimates returned to BRUCE differ
for the two codes. Using the code of Appendix A, the estimates of
A and S are calculated for both the number of fails and fires -- the
one obtained for the smaller of the preceding quantities being returned.
Actually, the choice is made in BRUCE after returning from FOREST.
A subroutine called SWAP is called when the number of fails is less
than the number of fires. In this event the number of fails will be
retained for further calculation. If the number of fires is less than
or equal to the number of fails no transfer to subroutine SWAP will
take place. Thus all calculations using the code in Appendix A
depend on the smaller of the two quantities, number of fails and
number of fires. This approach is advocated in Reference 4.

Using the code of Appendix B, the number of fires and fails
are equally weighted regardless of which is smaller. The basis
for this approach is described in Reference 3. Both the number
of fires and fails are used throughout the program. Subroutine
FOREST computes approximations to the mean and standard
deviation by first calculating estimates of these quantities based on
the number of fails and fires alone and then averaging them. These
averages are weighted so that the quantities calculated using the
smaller value between the number of fires or fails is more heavily
favored.

Thus using either code approximate estimates of A and S are
obtained using FOREST. After writing out the information read
in plus the starting values of S and A just chosen, a subroutine
entitled PROB is called. This subroutine calculates the probabili-
ties P_i and Q_i associated with each energy level H_i. The procedure
followed in this subroutine is a modification of an existing program

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used for calculating the probabilities of normally distributed random variables (Reference 2). The program which was modified is used to calculate probabilities from the normal probability integral defined as:

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-\frac{1}{2}t^2} dt .$$

The integral gives the probability that the normalized random variable t will be between -t and t on the normal probability curve shown in Figure 2. That is, the probability P(x) is equal to the area under the curve from -t to t. For small t the integral above can be evaluated from a power series:

$$P(x) = t \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[1 - \frac{t^2}{2 \cdot 1! \cdot 3} + \frac{t^4}{2^2 \cdot 2! \cdot 5} - \frac{t^6}{2^3 \cdot 3! \cdot 7} + \cdots \right].$$

For t > 2.90 a more efficient method of evaluating P(x) is achieved by using the asymptotic series defined as:

$$P(x) \cong 1 - \left(\frac{2}{\pi}\right) \frac{e^{-\frac{1}{2}t^2}}{t} \left(1 - \frac{1}{t^2} + \frac{1 \cdot 3}{t^4} - \frac{1 \cdot 3 \cdot 5}{t^6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{t^8} - \cdots \right).$$

For small t, (t < 1.20), the economy of calculating the normal probabilities is enhanced if use is made of telescoping the power series. For all terms beyond and including t^8, use is made of the telescoping effect produced by:

$$t^8 = \frac{1}{128} \left(256 t^2 - 160 t^4 + 32 t^2 - 1 + T8 \right)$$

which is deduced from Chebyshev polynomials (Reference 2). For the range 1.2 < t < 2.9 application of the original power series is satisfactory.

THE PROBABILITY THAT X LIES BETWEEN
-t AND t IS DEFINED AS:

$$P(X) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-\frac{1}{2}t^2} dt.$$
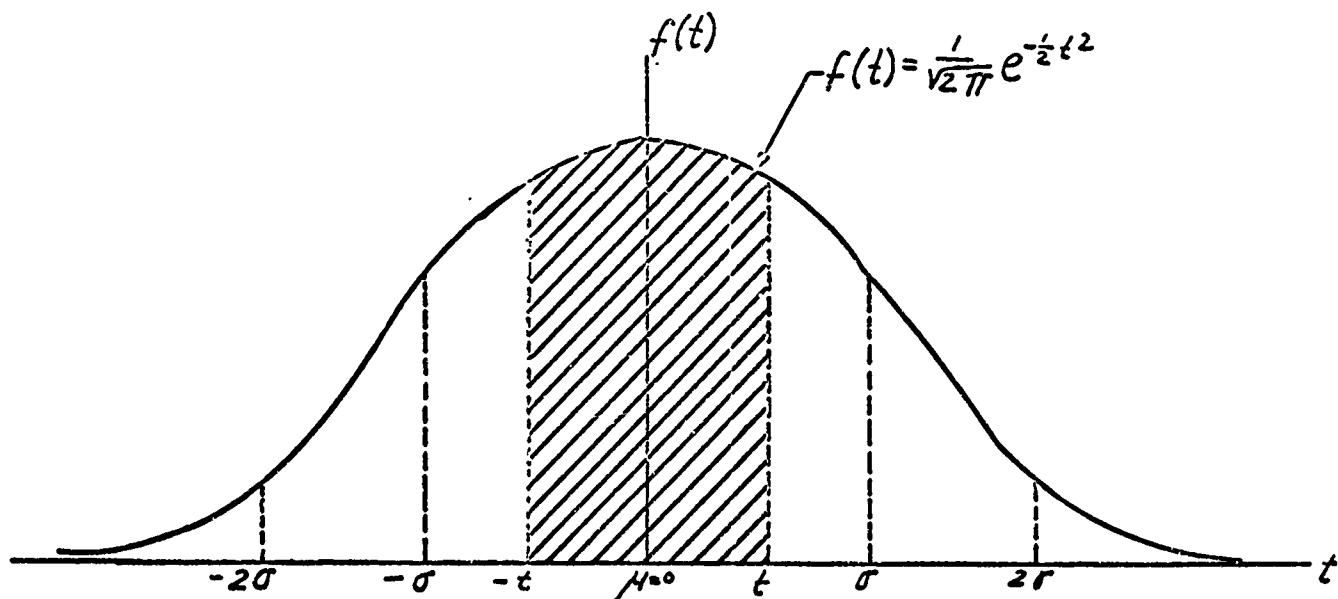
NORMAL FREQUENCY FUNCTION FOR THE
NORMALIZED RANDOM VARIABLE t.

FIGURE 2

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For each range, $t < 1.20$, $t > 2.90$, and $1.20 < t < 2.90$, probabilities were calculated and compared with those recorded in statistical tables. Because of the good agreement between the calculated and tabulated values and the relative ease with which a desired degree of accuracy was achieved, the approach using approximating series was employed.

However, as indicated, the method had to be modified. The reason for this is evident from Figure 3. It may be seen that the desired probability is equal to the area under the probability curve from $-\infty$ to t. Because of the symmetrical nature of the normal probability curve, the following technique was applicable. The area under the curve was obtained for the interval -t to t. This value was halved and either subtracted from 0.5 (for negative t) or added to 0.5 if t was positive. Thus, the probability for $-\infty$ to t was determined.

In addition, the subroutine PROB calculates the normal random variable associated with the random variable x from:

$$t = \frac{x - A}{S}$$

where A is the mean and S is the standard deviation. In terms of the symbols used in the code:

$$Y_i = \frac{H_i - A}{S}$$

The value of Z is adjusted too, using the expression:

$$Z_i = \frac{1}{\sqrt{2\pi}} e^{-\frac{(H_i - A)^2}{2S^2}}$$

each time PROB is called. This is necessary since $Z_i = f(A, S)$.

The corrections for S and A are calculated using the current values of P, Q, and Z as obtained from the subroutine PROB. The new values of S and A are used to evaluate $F_1(S, A)$ and $F_2(S, A)$ and then compare these with zero. If both are sufficiently close to zero, the order of accuracy may be determined by the User. For instance, 1×10^{-3} and 1×10^{-4} were chosen for $F_1(S, A)$ and $F_2(S, A)$, respectively, for the code shown in Appendix A while

THE PROBABILITY THAT X LIES BETWEEN
AND t IS DEFINED AS:

$$P(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}t^2} dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$
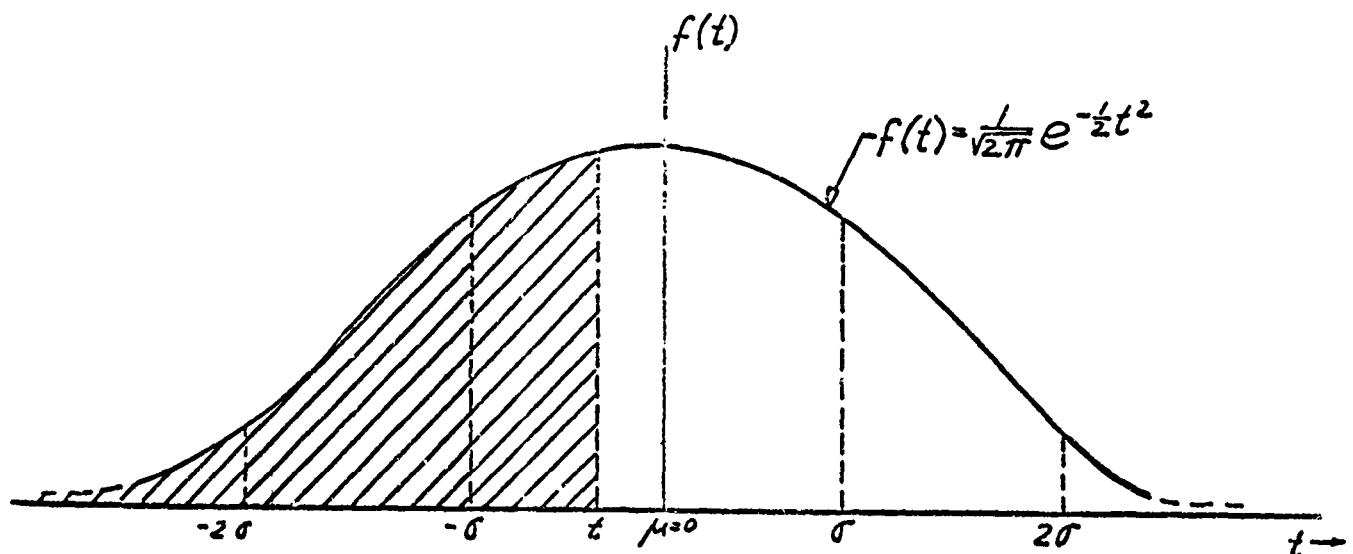
NORMAL FREQUENCY FUNCTION FOR THE
NORMALIZED RANDOM VARIABLE t

FIGURE 3

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1×10^{-4} and 1×10^{-5} were chosen for the code of Appendix B. The mean, standard deviation, $F_1(S, A)$ and $F_2(S, A)$ are then written out and transfer is made to subroutine SOLVE. This subroutine is used to calculate such statistical quantities as the standard error of the mean, standard error of the standard deviation, the sure-fire probability point and a corresponding safe or NO-FIRE probability level. If the values of $F_1(S, A)$ and $F_2(S, A)$ are not sufficiently close to zero, however, the iterations are continued so that new corrections for A and S are obtained. This process of iteration is allowed to continue until $F_1(S, A)$ and $F_2(S, A)$ become as small as desired or until the number of iterations reach some predetermined value chosen by the User (1,000 iterations were chosen in the code shown in Appendix A, 100 for the code shown in Appendix B).

The values of $F_1(S, A)$, $F_2(S, A)$ the next to the last values of S and A, the final values of S and A and the number of iterations are written out for each run. This is done so that if all 1,000 (or 100) iterations are carried out because the error in either $F_1(S, A)$ or $F_2(S, A)$ is larger than the bound specified for each, one is able to observe two successive values of S and A and decide whether convergence has actually been achieved. The results as computed by each code for three tests are shown at the end of the respective code. Note that for these three tests only two, three or four iterations were required. This rapid convergence is due to the good approximation of the mean and standard deviation obtained using subroutine FOREST.

Once the maximum likelihood estimates of A and S are approximated, subroutine SOLVE is called for purposes of estimating the standard errors associated with A and S and calculating the sure-fire and safe or NO-FIRE energy levels. To estimate the standard errors two parameters G and H must be determined. These are defined by:

$$G^2 = \sum_i W_i \Big/ \sum_i W_i \left[\frac{z_{i-1}^2}{q_{i-1}^2} + \frac{z_i^2}{p_i^2} \right],$$

and

$$H^2 = \sum_i W_i \Big/ \sum_i W_i \left[\frac{x_{i-1}^2 z_{i-1}^2}{q_{i-1}^2} + \frac{x_i^2 z_i^2}{p_i^2} \right],$$

where:

$$W_i = \prod_{j=0}^{j-1} \frac{Q_j}{P_j} \quad , \quad i > 0$$

$$W_o = 1$$

$$W_i = \prod_{j=-1}^{i} \frac{P_j}{Q_j} \quad , \quad i < 0 .$$

Subroutine SOLVE makes use of the definitions of G and H and calculates these two quantities directly. Furthermore it calculates the high and low probability points and the other secondary statistics (standard errors).

The standard error of the mean is calculated from $\sigma_m = SG/\sqrt{N}$, the standard error of the standard deviation from $\sigma_s = SH/\sqrt{N}$, the standard error at the P percent energy level from $\sigma_p = S \sqrt{G^2 + K^2 H^2}/\sqrt{N}$. In the above, N is the total number of fires and K is a constant associated with a particular percent point. The sure-fire and safe or NO-FIRE energy levels are then calculated from:

$$\text{Sure-Fire} = A + KS \tag{11}$$
$$\text{NO FIRE} = A - KS \tag{12}$$

for the 50% confidence band. To add confidence to these reliability points, adjustment to Equation (11) and (2) are made using Sure-Fire $= \text{Sure-Fire} + C \dfrac{N + C'}{N} \sigma_p$ and NO-FIRE $= \text{NO-FIRE} - C \dfrac{N + C'}{N} \sigma_p$

where C and C' are constants associated with a particular confidence interval. That is, the estimates of the sure-fire and NO-FIRE energy levels should be correct P percent of the time or the risk of being wrong is anticipated for only 1-P percent of the time (P = 90, 95, 99,...).

As stated earlier, the code in Appendix A computes all quantities based on the smaller value between the number of fires or fails obtained in the Bruceton test. On the other hand, the code in Appendix B computes all quantities without regard to which value is smaller except in weighing averages in which case the smaller is more heavily favored.

Finally, the main program writes out the desired statistical values

Regardless of which code is used the data cards are arranged identically. The following describes this arrangement:

Card 1 -- Six fields of 10 columns each are reserved. Columns 1-10 and 11-20 are reserved for constants associated with the reliability and confidence level, respectively. Columns 21-30 and 31-40 are reserved for specifying the reliability and confidence level, respectively. Columns 41-50 are reserved for specifying the NO-FIRE energy level. Columns 51-60 are reserved for a second constant associated with the confidence level.

Card 2 -- Four fields, one of 20 columns, one of 5 columns and 2 of 10 columns each are reserved. The first field is used for identification. The second contains the number of energy levels in the test. The third and fourth contain the lowest energy level in log units and the difference between successive energy levels (step interval) in log units, respectively.

Card 3 -- Twelve fields, alternating between eight and four columns each are reserved. The fields of eight columns each contain the energy levels in ascending order. The corresponding number of fires at each energy level is specified in the right adjacent field of four columns.

Card 4 -- Six fields of 10 columns each are reserved. The number of fails at each energy level -- starting at the lowest energy level -- is placed in these fields from left to right.

Table II on Page 52 and Page 20 and 21 of Reference 4 contain the constants associated with several reliability and confidence levels.

This example will demonstrate the input data cards for an actual set of test data. The table below shows the typical results of a Bruceton test. It is desired to calculate the 99.9% sure-fire and safe-fire reliability levels at the 95% confidence level:

Firing Current	Log of Firing Current	Test Level	Number of Fires	Number of Fails
374	2.57287	1	0	4
392	2.59329	2	4	10
410	2.61278	3	10	10
429	2.63246	4	9	2
450	2.65321	5	1	0

From this table and Reference 4 the data cards for input may be prepared.

Card 1
Card Columns

1	1011	2021	3031	4041	5051	6061-80
"3.090"	"1.96"	"99.9"	"95.0"	"0.1"	"1.2"	

Card 2
Card Columns

1	2021	2526	3536	4546	80
"M2 Squib Test D"	"5"	"2.57287"	"0.02"		

Card 3
Card Columns

1	89	1213	2021	2425	3233	3637	4445	4849	5657	6061
"374."	"0."	"393."	"4."	"410."	"10."	"429."	"19."	"450."	"1."	

Card 4
Card Columns

1	1011	2021	3031	4041	5051	6061	80
"4."	"10."	"10."	"2."	"0."			

The output using these data cards was obtained for both codes and is shown at the end of each (Appendix A and B).

The code of Appendix B appears to be superior. First, both the number of fails and numbers of fires are utilized in computing the various statistical quantities. Thus, a larger effective sample size is employed which for non-exact measurements is advantageous. Also, examination of the results at the end of each code shows that a higher ALL-FIRE and lower NO-FIRE are obtained using Code B if the number of fires and fails differ. This is desirable for increased reliability and safey. If the number of fires and fails are equal, the results using either code are identical. Consequently, for the two codes use of that of Appendix B is advocated.

In certain instances, the estimates of A and S from the first two moments (approximation method) are poor approximations to the maximum likelihood values. In particular when the step size between test levels becomes large (d > 2S) and when small sample sizes are used (<50) the approximations to A and S become poor -- that is, the maximum likelihood estimates of A and S are not well approximated by the first two moments.

If a difference in the two estimates does exist, the maximum likelihood quantities should be used because they are expected to be better than those generated by the first two moments. Furthermore, the standard errors of the mean and standard deviation are based on their respective maximum likelihood estimates and will not be applicable to the approximation values when a large difference exists. This means that the calculated extremes (sure-fire and NO-FIRE energy levels) will be poor estimates of the true quantities as well.

To determine the effect of sample size on the estimates of the standard deviation, a series of tests were conducted using sample sizes of 20, 30, 50 and 100 items. The mean and standard deviation for each sample were calculated by estimating the maximum likelihood quantities and by utilizing the first two moments. Because the Bruceton method is constructed to provide test data near the mean, that quantity is nearly always a good estimate of the population value. However, the standard deviation is not so well determined. Therefore, in what follows, special attention is paid to the effect of sample size on the standard deviation only.

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Appendix C shows calculated values of the mean and standard deviation as determined by the maximum likelihood equations and from the first two moments. It is immediately obvious that little difference exists between the means as calculated by either method. But for small samples the standard deviation as calculated from the first two moments differs quite appreciably from that obtained utilizing the maximum likelihood equations. As stated, standard errors are based on the maximum likelihood quantities. In turn, the extremes of the probability distribution are likewise influenced because of their dependence on the mean, standard deviation and associated standard errors.

It is apparent that samples containing over 100 items should be used. The use of maximum likelihood theory for the cases in which the sample size is less than 100 is equally advocated. Specifically, Table 9 shows that for samples of 20 and 30 items large differences between the standard deviations occur frequently. For samples of size 50, better precision is achieved. Nevertheless, some poor results are still evident. For samples of size 100 even closer agreement between the two methods of calculation is achieved and it is anticipated that larger sample sizes would produce results differing only slightly from one method to the other.

It should be pointed out that the good results in the 50 item samples were due primarily to a knowledge of the past history of the test vehicle. However, the usual situation is one in which little or no data is available for the item since it is the purpose of these tests to generate such data. In these cases it is imperative to use large randomly chosen samples or if this is not practical, the maximum likelihood equations should be applied in calculating statistical quantities.

An example will best illustrate the effect of using small sample data in calculating the extremes of a population. For a sample of size 30, these were the results:

	Maximum Likelihood Equations	Approximate Formulae
Mean	2.20095909	2.18977
Standard Deviation	0.18969359	0.14878
Standard Error of Mean	0.041085	0.03283
Standard Error of Standard Deviation	0.073707	0.05342
Sure-Fire Level	3.27356932	2.78843
NO-FIRE Level	1.12834880	1.59111

These values are recorded in log volts. If the sure-fire and NO-FIRE levels are converted to volts, the potential danger becomes obvious -- that is, using maximum likelihood equations the sure-fire level is about 1,877 volts as against 615 volts using the approximation, and the corresponding NO-FIRE levels are 13.5 volts and 39 volts, the former value was calculated from the maximum likelihood equations. Consequently, within the probability limits indicated, the sure-fire and NO-FIRE levels differ roughly by a factor of 3. Practically, this means that for this case in which a firing voltage of about 1,877 volts is required to provide sure-fire conditions, only 615 volts might be made available by a User relying on approximate formulas only. For similar reasons, safe voltage levels may be inaccurately specified. Thus, anticipated reliabilities are not closely achieved.

Although the discrepancies between the two methods become less pronounced with increasing sample size, maximum likelihood methods should be employed in view of their assumed superiority and the availability of a digital computer algorithm.

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REFERENCES

1. Abraham Golub, Frank E. Grubbs, <u>Analysis of Sensitivity Experiments When the Levels of Stimulus Cannot Be Controlled,</u> American Statistical Association Journal, Vol. 51, pp. 257-265, June 1956.

2. Daniel D. McCracken, W.S. Dorn, <u>Numerical Methods and FORTRAN Programming,</u> John Wiley and Sons, Inc., New York, New York, 1964.

3. Forrest L. McMains, <u>A Digital Computer Program for a Bruceton Analysis,</u> Picatinny Arsenal Technical Memorandum 1254, August 1963.

4. John W. Tukey et al, <u>Statistical Analysis for a New Procedure in Sensitivity Experiments,</u> Applied Mathematics Panel National Defense Research Committee Report 101.1R, July 1944.

APPENDICES

APPENDIX A

FORTRAN CODE FOR DETERMINING THE MEAN AND STANDARD DEVIATION
FROM THE SOLUTION OF THE MAXIMUM LIKELIHOOD EQUATIONS BY
MEANS OF THE METHOD OF SUCCESSIVE APPROXIMATIONS
(METHOD A)

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```
C     BRUCE STATIS MAX LIKE ESTIMATORS
      DIMENSION H(99),        Z(99),P(99),Q(99),XN(99),X(99),Y(99),W(99)
      DIMENSION  DZS(99), DZA(99),DQS(99),DQA(99),DPS(99),DPA(99)
      DIMENSION XNF(99),ANTH(50),IDENT(20)
      COMMON  GSQRD , HSQRD , SIGSQA , SIGSQS , SIGMAP , FIR50
      COMMON FNOT50,FIREK,FNOTK
C     I=NO. OF ENERGY LEVELS , S=STEP SIZE(DIST. BETWEEN 2 ENERGY LEVELS
C     A=MIDDLE ENERGY LEVEL(IF I IS ODD) OR AVG. OF MIDDLE 2 ENERGY LEVE
C     1IS EVEN)
    1 READ  (5      ,2)      RELK,CONK,RELV,CONLV,XRELV,CST
    2 FORMAT( 6F10.0 )
      CALL FOREST (XN,XNF,I,ANTH,AFIRE,AFAIL,SFIRE,IDENT,SFAIL )
      DO 7855 J = 1,I
 7855 H(J) = ALOG10(ANTH(J))
C     H(J)=ENERGY LEVELS , XN(J)=NO. OF FIRES AT A GIVEN H(J) , Y(J)=NOR
C     1RANDOM VARIABLE , X(J)=ABS. VALUE OF Y(J) , P(J)=PROB. OF FIRE ,
C     Q(J)=PROB. OF NO FIRE , Z(J)=DERIVATIVE OF P(J)
      SUMN = 0.0
      SUMNF = 0.0
      DO 1216 J = 1,I
      SUMN = SUMN +XN(J)
 1216 SUMNF = SUMNF +XNF(J)
      IF (SUMN - SUMNF ) 8571,8571,1758
 8571 A = AFIRE
      S = SFIRE
      GO TO 8578
 1758 A = AFAIL
      S = SFAIL
 8578 IF ( SUMN.GT.SUMNF ) CALL SWAP (XN,XNF,I )
   24 WRITE (6,100 )(IDENT(J),J=1,20 )
  100 FORMAT(1H1/1X,20A2/30X,66HCALC. OF MEAN AND STD. DEVIATION BY SOLN
     1. OF MAX. LIKELIHOOD EQNS.//)
      WRITE (6      ,10)
   10 FORMAT(40X,12HENERGY LEVEL,25X,18HNO. OF FIRES/FAILS//)
      WRITE (6,11 )(H(J),XN(J),J=1,I)
   11 FORMAT(40X,F10.5,30X,F4.0//)
      WRITE (6      ,1010)A,S,I
 1010 FORMAT(50X,7HAVG. = ,F10.5//50X,17HAPPRX STND DEV = ,F10.5//50X,23
     1HNO. OF ENERGY LEVELS = ,I4// )
C     COUNT OF THE NO. OF ITERATIONS
      ITCNT = 0
    6 CALL PROB (H,X,Y,Z,A,S,P,Q,I)
C     BEGIN SUCCESSIVE APPROXIMATIONS(NEWTON RAPHSON METHOD)
C     DERIVATIVE OF Z WITH RESPECT TO S AND A
C     DERIVATIVE OF Q WITH RESPECT TO S AND A
C     DERIVATIVE OF P WITH RESPECT TO S AND A
      DO 436 J=1,I
      DZS(J)=((H(J)-A)**2-S*S)/S**3*Z(J)
      DZA(J)=(H(J)-A)/(S*S)*Z(J)
      DQS(J)=-(H(J)-A)/(Q(J)*S)**2*Z(J)
      DQA(J)=-Z(J)/(Q(J)**2*S)
      DPS(J)=(H(J)-A)/(P(J)*S)**2*Z(J)
      DPA(J)=Z(J)/(P(J)*P(J)*S)
  436 CONTINUE
      SUM1=0.0
```

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```
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      SUM6=0.0
      SUM=0.0
      SSUM=0.0
      DO 555 J=2,I
      SUM1=SUM1+XN(J)*(DZS(J-1)/Q(J-1)+DQS(J-1)*Z(J-1)-DZS(J)/P(J)-DPS(J
     1)*Z(J))
      SUM2=SUM2+XN(J)*(Z(J-1)/Q(J-1)-Z(J)/P(J))
      SUM3=SUM3+XN(J)*(DZA(J-1)/Q(J-1)+DQA(J-1)*Z(J-1)-DZA(J)/P(J)-DPA(J
     1)*Z(J))
      SUM4=SUM4+XN(J)*((H(J-1)-A)*DZS(J-1)/Q(J-1)+(H(J-1)-A)*CQS(J-1)*Z(
     1J-1)-(H(J)-A)*DZS(J)/P(J)-(H(J)-A)*DPS(J)*Z(J))
      SUM5=SUM5+XN(J)*((H(J-1)-A)*Z(J-1)/Q(J-1)-(H(J)-A)*Z(J)/P(J))
      SUM6=SUM6+XN(J)*(H(J-1)*DZA(J-1)/Q(J-1)+H(J-1)*DQA(J-1)*Z(J-1)-Z(J
     1-1)/Q(J-1)-A/Q(J-1)*DZA(J-1)-A*Z(J-1)*DQA(J-1)-H(J)/P(J)*DZA(J)-DP
     2A(J)*H(J)*Z(J)+Z(J)/P(J)+A*Z(J)*DPA(J)+A/P(J)*DZA(J))
      SUM=SUM+XN(J)*(Z(J-1)/Q(J-1)-Z(J)/P(J))
      SSUM=SSUM+XN(J)*((H(J-1)-A)*Z(J-1)/Q(J-1)-(H(J)-A)*Z(J)/P(J))
  555 CONTINUE
C     DERIVATIVE OF F1(S,A) WITH RESPECT TO S AND A
C     DERIVATIVE OF F2(S,A) WITH RESPECT TO S AND A
C     FAS=F1(S,A) , SAS=F2(S,A)
      DFDS=1.0/S*SUM1-1.0/S**2 *SUM2
      DFDA=1.0/S*SUM3
      DSDS=1.0/S**2 *SUM4-2.0/S**3 *SUM5
      DSDA=1.0/S**2 *SUM6
      FAS=1.0/S*SUM
      SAS=1.0/S**2*SSUM
  303 D=DFDS*DSDA-DFDA*DSDS
C     CORRECTION FOR S
      XH=(DFDA*SAS-DSDA*FAS)/D
C     CORRECTION FOR A
      XK=(DSDS*FAS-DFDS*SAS)/D
C     CORRECTED A
      AA=A+XK
C     CORRECTED S
      SS=S+XH
   17 S=SS
      A=AA
C     VALUE OF F1(A,S) AND F2(A,S) OBTAINED FROM FINAL A AND S
      ZERO1=0.0
      ZERO2=0.0
      DO 69 J=2,I
      ZERO1=ZERO1+XN(J)*(Z(J-1)/Q(J-1)-Z(J)/P(J))
   69 ZERO2=ZERO2+XN(J)*((H(J-1)-A)*Z(J-1)/Q(J-1)-(H(J)-A)*Z(J)/P(J))
      IF(ABS(ZERO1)-1.E-3) 68,68,15
   68 IF(ABS(ZERO2)-1.E-4) 16,16,15
   15 ITCNT = ITCNT + 1
      IF(ITCNT - 1000 ) 151,16,16
  151 GO TO 6
   16 WRITE (6      ,6666)ZERO1,ZERO2,A,AA,S,SS
 6666 FORMAT(50X,8HZERO1 = ,E20.8//50X,8HZERO2 = ,E20.8//50X,9HNLMEAN =
     1,F10.8//50X,7HMEAN = ,F10.8//50X,13HNLSTND. DEV. = ,F10.8//50X,
```

```
    221HSTANDARD DEVIATION = ,F15.8//)
       WRITE (6       ,111)ITCNT
 111 FORMAT(50X,20HNO. OF ITERATIONS = ,I5//)
       CALL SOLVE(P,Q,XN,Y,Z,I,A,S,H,RELK,CONK,CST)
       WRITE (6       ,77)
  77 FORMAT(40X,49HCALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS//)
       WRITE (6       ,971)GSQRD,HSQRD,SIGSQA,SIGSQS,SIGMAP,FIRSO,    FNOT50
      1,RELV,CONLV,FIREK,XRELV,CONLV,FARTK
 971 FORMAT(50X,11HGSQUARED = ,F15.6//50X,11HHSQUARED =,F10.6//50X,29HS
      1TAND. ERROR OF MEAN SQRD. = ,F10.6//50X,36HSTAND. ERROR OF STAND.
      2DEV. SQRD. = ,F10.8//50X,35HSTAND. ERROR AT P PERCENT HEIGHT = ,F1
      30.8//50X,33H99.9FIRE AT 5OCONFIDENCE LEVEL = ,F10.8//50X,32HO.1FIR
      4E AT 5OCONFIDENCE LEVEL = ,F10.8//50X,F6.2,7HFIRE AT,F6.2,19HCONFI
      5DENCE LEVEL = ,F10.8//50X,F6.2,7HFIRE AT,F6.2,19HCONFIDENCE LEVEL
      6= ,F10.8// )
       GO TO 1
       END
```

```
      SUBRØUTINE PRØB(       H,X,Y,Z,A,S,P,Q,I)
      DIMENSIØN         .          H(99),X(99),Y(99),Z(99),P(99),Q(99)
      DØ 7 J = 1, I
      HMASQ = (H(J)-A)**2
      EXPNT = HMASQ/(2.*S*S)
      Z(J) = (1./2.5J663)/EXP(EXPNT)
      Y(J) = (H(J)-A)/S
      X(J) = ABS(Y(J))
      IF (X(J)-1.20 ) 21,21,22
   21 XSQ = X(J)*X(J)   :
      P(J)=0.79788455*X(J)*(0.99999774-XSQ*(0.16659433-XSQ*(.24638310E-0
     11-XSQ*.23974867E-02)))  - ..
      IF ( Y ( J ) ) 400, 401, 402
  400 P ( J ) = 0.5 - P ( J ) / 2.0
      GØ TØ 403
  401 P ( J ) = 0.5
      GØ TØ 403
  402 P(J)=P(J)/2.0+0.5
  403 Q (J ) = 1.0- P ( J )
      GØ TØ 7
   22 IF (X(J) - 2.90 ) 25,26,26
   25 XSQ = X(J) * X(J)
      P(J) = 1.0
      PT = 1.0
      FACT = 1.0
      ØDD = 3.0
   29 PT = -PT*XSQ/(2.0*FACT )
      T = PT/ØDD
      P(J) = P(J) + T
      IF (ABS(T) - 0.00007 ) 63 ,70 ,70
   70 FACT = FACT + 1.0
      ØDD = ØDD + 2.0
      GØ TØ 29
   63 P(J) = 0.79788455*X(J)*P(J)
      IF ( Y ( J ) ) 500, 501, 502
  500 P ( J ) = 0.5 - P ( J ) / 2.0
      GØ TØ 503
  501 P ( J ) = 0.5
      GØ TØ 503
  502 P(J)=P(J)/2.0+0.5
  503 Q (J ) = 1.0- P ( J )
      GØ TØ 7
   26 RXSQ = 1.0/(X(J)*X(J))
      P(J) = 1.0-0.79788455*EXP(-X(J)*X(J)/2.0)/X(J)*(1.0-RXSQ*(1.0-RXSQ
     1*   (3.0-RXSQ*(15.0-RXSQ*105.0))))
      IF ( Y ( J ) ) 600, 601, 602
  600 P ( J ) = 0.5 - P ( J ) / 2.0
      GØ TØ 603
  601 P ( J ) = 0.5
      GØ TØ 603
  602 P(J)=P(J)/2.0+0.5
  603 Q (J ) = 1.0- P ( J )
    7 CØNTINUE
      RETURN
      END
```

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```
      SUBRØUTINE FØREST(FIX,FAX,N,X,UFIX,UFAX,VFIX,IDENT,VFAX )
      ALCG(W)=.4343*ALØG(W)
      ANT(W)=EXP(W/.4343)
      DIMENSIØNX(5J),IDENT(20),FIX(99),FAX(99),S (30,15)
      DATA  S                 /64*0.,25.,40.,43.,46.,49.,51.,16*0.,2.
     15.,27.,31.,35.,37.,42.,44.,47.,49.,51.,12*0.,25.,26.,28.,29.,31.,3
     24.,36.,39.,40.,43.,46.,48.,49.,51.,9*0.,25.,27.,28.,31.,32.,34.,36
     3.,37.,39.,40.,41.,42.,44.,46.,48.,49.,51.,4*0.,25.,26.,27.,27.,28.
     4,30.,30.,31.,32.,33.,35.,36.,37.,39.,41.,42.,43.,44.,46.,49.,49.,5
     51.,25.,26.,27.,27.,28.,29.,30.,30.,31.,32.,33.,34.,35.,36.,37.,38.
     6,40.,41.,42.,43.,45.,46.,47.,49.,51.,52.,25.,25.,26.,27.,29.,29.,3
     70.,30.,32.,32.,33.,34.,35.,36.,38.,39.,40.,42.,43.,44.,45.,46.,47.
     8,49.,51.,52.,25.,26.,27.,27.,29.,29.,30.,31.,32.,33.,34.,35.,36.,3
     97.,38.,39.,40.,41.,43.,44.,45.,47.,49.,50.,51.,53.,26.,26.,27.,28.
     A,29.,30.,31.,31.,32.,34.,35.,36.,37.,39.,39.,40.,41.,42.,44.,44.,4
     85.,47.,49.,50.,52.,53.,26.,27.,28.,28.,29.,31.,31.,33.,34.,34.,36.
     C,37.,38.,39.,40.,41.,42.,43.,44.,45.,46.,47.,49.,51.,52.,53.,26.,2
     07.,28.,29.,30.,31.,32.,33.,34.,35.,36.,37.,38.,39.,40.,41.,42.,43.
     E,44.,45.,46.,47.,49.,50.,52.,54./
      READ (5,639)(IDENT(J),J=1,20),N,C,D
 639  FØRMAT(20A1,I5,2F10.0)
      READ(5,637) (X(I),FIX(I),I=1,N)
 637  FØRMAT(6(F8.0,F4.0))
      READ(5,638) (FAX(I),I=1,N)
 638  FØRMAT( 6F10.0  )
      AFIX=0.0
      AFAX = 0.0
      BFIX=0.0
      BFAX=0.0
      SFIX=0.0
      SFAX=0.0
      I=0
3010  W=I
      AFIX=AFIX+W*FIX(I+1 )
      AFAX=AFAX+W*FAX(I+1)
      BFIX = BFIX + (W**2 ) * FIX(I+1)
      BFAX=BFAX+(W**2)*FAX(I+1)
      SFIX=SFIX+FIX(I+1)
      SFAX=SFAX+FAX(I+1)
      I=I+1
      IF(I-N+1)3010,3010,1115
1115  R = D
      TFIX=((SFIX*BFIX)-AFIX**2)/(SFIX**2)
      TFAX=((SFAX*BFAX)-AFAX**2)/(SFAX**2)
  13  UFIX=C+R*(AFIX/SFIX-0.5)
      UFAX=C+R*(AFAX/SFAX+0.5)
      IF(TFIX-0.3) 118,23,23
 118  CIX=TFIX*100.
      LIX=CIX
4111  WEEIX = 500.
      I=0
 112  WEEIX = AMIN1(WEEIX,ABS(UFIX-ALCG(X(I+1))))
      I=I+1
      IF(I-N+1)112,112,113
 113  RIX=ABS(WEEIX/R)
```

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```
      KIX = IFIX (RIX * 22. ) + 1
      VFIX=( S(LIX,KIX)/100.)*R
 2424 IF(TFAX-0.3)117,2525,2525
  117 CAX=TFAX*100.
      LAX=CAX
  114 WEEAX=500.
      I=0        .
  115 WEEAX = AMIN1(WEEAX,ABS(UFAX-ALCG(X(I+1))))
      I=I+1
      IF(I-N+1)115,115,116
  116 RAX=ABS(WEEAX/R)
      KAX = IFIX (RAX*22.)+1
      VFAX =.( S(LAX,KAX)/100.)*R
      GØ TØ 26
   23 VFIX=1.62*R*(TFIX+0.029)*SQRT(SFIX/(SFIX-1.))
      GØ TØ 2424.
 2525 VFAX=1.62*R*(TFAX+0.029)*SQRT(SFAX/(SFAX-1.))


   26 RETURŃ
      FND
```

```
       Subroutine SWAP (XN, XNF, I)

       Dimension XN(99), XNF(99)

       DØ    4143    J = 2, I

4143   XN(J) = XNF (J-1)

       RETURN

       END
```

```
      SUBRØUTINE SØLVE (P,Q,XN,Y,Z,I,A,S,H,RELK,CØNK,CST )
      CØMMØN GSQRD,HSQRD,SIGSQA,SIGSQS,SIGMAP,FIR5O, FNØT5O,FIREK,FNØTK
      DIMENSION H(99),P(99),Q(99),XN(99),Y(99),Z(99),W(99)
      SUMNJ = 0.0
      W(1) = 1.0
      WJSUM = 0.0
      GSUM = 0.0
      HSUM = 0.0
      DØ 51 J=2,I
   51 W(J)=Q(J-1)/P(J-1)*W(J-1)
      DØ 61 J=1,I
      SUMNJ = SUMNJ + XN(J)
   61 WJSUM = WJSUM + W(J)
      DØ 741 J=2,I
      GSUM = GSUM + W(J-1)*(Z(J-1)**2/Q(J-1)**2+Z(J)**2/P(J)**2)
  741 HSUM = HSUM+W(J-1)*(Y(J-1)**2*Z(J-1)**2/Q(J-1)**2+Y(J)**2*Z(J)**2/
     1P(J)**2)
      GSQRD = WJSUM/GSUM
      HSQRD = WJSUM/HSUM
      SIGSQA=S*S/SUMNJ*GSQRD
      SIGSQS = S*S/SUMNJ*HSQRD
      SIGMAP = SQRT(SIGSQA+RELK**2*SIGSQS )
      FIR5O = A+RELK*S
      FNØT5O = A-RELK*S
      FIREK = FIR5O+CØNK*((SUMNJ+CST)/SUMNJ)*SIGMAP
      FNØTK = FNØT5O-CØNK*((SUMNJ+CST)/SUMNJ)*SIGMAP
      RETURN
      END
```

M2 Squib Test D

CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.

ENERGY LEVEL	NO. OF FIRES/FAILS
2.57287	0.
2.59329	4.
2.61278	10.
2.63246	9.
2.65321	1.

AVG. = 2.60870

APPRX STND DEV = 0.02159

NO. OF ENERGY LEVELS = 5

ZERO1 = -0.52526593E-04

ZERO2 = 0.28066803E-04

NLMEAN = 2.60874474

MEAN = 2.60874474

NLSTND. DEV. = 0.02079978

STANDARD DEVIATION = 0.02079978

NO. OF ITERATIONS = 2

CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS

GSQUARED = 0.696013

HSQUARED = 0.791929

STAND. ERROR OF MEAN SQRD. = 0.00001234

STAND. ERROR OF STAND. DEV. SQRD. = 0.00001428

STAND. ERROR AT P PERCENT HEIGHT = 0.01220044

99.9FIRE AT 50CONFIDENCE LEVEL = 2.67301604

0.1FIRE AT 50CONFIDENCE LEVEL = 2.54447341

99.9FIRE AT 95.00CONFIDENCE LEVEL = 2.69812453

0.10FIRE AT 95.00CONFIDENCE LEVEL = 2.51936489

CALC. ØF MEAN AND STD. DEVIATIØN BY SØLN. ØF MAX. LIKELIHØØD EQNS.

ENERGY LEVEL	NØ. ØF FIRES/FAILS
0.14301	0.
0.16435	2.
0.18469	9.
0.20683	10.
0.22531	3.

AVG. = 0.18602

APPRX STND DEV = 0.02279

NØ. ØF ENERGY LEVELS = 5

ZERØ1 = 0.19371510E-05

ZERØ2 = 0.22351742E-07

NLMEAN = 0.18685731

MEAN = 0.18685731

NLSTND. DEV. = 0.02369443

STANDARD DEVIATIØN = 0.02369443.

NØ. ØF ITERATIØNS = 3

CALCULATIØN ØF ALL-FIRE AND NØ-FIRE ENERGY LEVELS

GSQUARED = 0.846374

HSQUARED = 1.498057

STAND. ERRØR ØF MEAN SQRD. = 0.00001980

STAND. ERRØR ØF STAND. DEV. SQRD. = 0.00003504

STAND. ERRØR AT P PERCENT HEIGHT = 0.01882551

99.9FIRE AT 50CØNFIDENCE LEVEL = 0.26007312

0.1FIRE AT 50CØNFIDENCE LEVEL = 0.11364149

99.90FIRE AT 95.00CØNFIDENCE LEVEL = 0.29881602

0.10FIRE AT 95.00CØNFIDENCE LEVEL = 0.07489859

M6 Cap Test E

CALC. ØF MEAN AND STD. DEVIATIØN BY SØLN. ØF MAX. LIKELIHØØD EQNS.

ENERGY LEVEL	NØ. ØF FIRES/FAILS
2.43993	0.
2.42488	2.
2.44091	9.
2.45484	12.
2.46982	2.

AVG. =   2.44130

APPRX STND DEV =   0.01477

NØ. ØF ENERGY LEVELS =   5

ZERØ1 =   0.26762486E-04

ZERØ2 =   0.75111166E-06

NLMEAN = 2.44135469

MEAN = 2.44135469

NLSTND. DEV. = 0.01453041

STANDARD DEVIATIØN = 0.01453041

NØ. ØF ITERATIØNS =   2

CALCULATIØN ØF ALL-FIRE AND NØ-FIRE ENERGY LEVELS

GSQUARED =   0.785916

HSQUARED = 1.171202

STAND. ERRØR ØF MEAN SQRD. = 0.00000663

STAND. ERRØR ØF STAND. DEV. SQRD. = 0.00000989

STAND. ERRØR AT P PERCENT HEIGHT = 0.01005381

99.9FIRE AT 50CØNFIDENCE LEVEL = 2.48625368

0.1FIRE AT 50CØNFIDENCE LEVEL = 2.39645568

99.90FIRE AT 95.00CØNFIDENCE LEVEL = 2.50690502

0.10FIRE AT 95.00CØNFIDENCE LEVEL = 2.37580431

APPENDIX B

FORTRAN CODE FOR DETERMINING THE MEAN AND STANDARD DEVIATION
FROM THE SOLUTION OF THE MAXIMUM LIKELIHOOD EQUATIONS BY
MEANS OF THE METHOD OF SUCCESSIVE APPROXIMATIONS
(METHOD B)

```
C      BRUCE STATIS MAX LIKE ESTIMATORS
       DIMENSION H(99),        Z(99),P(99),Q(99),XN(99),X(99),Y(99),W(99)
       DIMENSION  DZS(99), DZA(99),DQS(99),DQA(99),DPS(99),DPA(99)
       DIMENSION XNF(99),ANTH(50),IDENT(20)
       COMMON  GSQRD  , HSQRD  , SIGSQA , SIGSQS , SIGMAP , FIR50
       COMMON FNOT50,FIREK,FNOTK
C      I=NO. OF ENERGY LEVELS , S=STEP SIZE(DIST. BETWEEN 2 ENERGY LEVELS
C      A=MIDDLE ENERGY LEVEL(IF I IS ODD) OR AVG. OF MIDDLE 2 ENERGY LEVE
C      1IS EVEN)
     1 READ (5      ,2)        RELK,CONK,RELV,CONLV,XRELV,CST
     2 FORMAT( 6F10.0 )
       CALL FOREST(XN,XNF,I,ANTH,A,S,IDENT )
       DO 7855 J = 1,I
  7855 H(J) = ALOG10(ANTH(J))
C      H(J)=ENERGY LEVELS , XN(J)=NO. OF FIRES AT A GIVEN H(J) , Y(J)=NOR
C      1RANDOM VARIABLE , X(J)=ABS. VALUE OF Y(J) , P(J)=PROB. OF FIRE ,
C      Q(J)=PROB. OF NO FIRE , Z(J)=DERIVATIVE OF P(J)
    24 WRITE (6,100 )(IDENT(J),J=1,20 )
   100 FORMAT(1H1/1X,20A2/30X,66HCALC. OF MEAN AND STD. DEVIATION BY SOLN
      1. OF MAX. LIKELIHOOD EQNS.//)
       WRITE (6      ,10)
    10 FORMAT(40X,12HENERGY LEVEL,25X,18HNO. OF FIRES/FAILS//)
       WRITE (6,11 ) (H(J),XN(J),XNF(J), J = 1,I )
    11 FORMAT(40X,F10.5,33X,F4.0,3X,F4.0//)
       WRITE (6      ,1010)A,S,I
  1010 FORMAT(50X,7HAVG. = ,F10.5//50X,17HAPPRX STND DEV = ,F10.5//50X,23
      1HNO. OF ENERGY LEVELS = ,I4// )
C      COUNT OF THE NO. OF ITERATIONS
       ITCNT = 0
     6 CALL PROB (H,X,Y,Z,A,S,P,Q,I)
C      BEGIN SUCCESSIVE APPROXIMATIONS(NEWTON RAPHSON METHOD)
C      DERIVATIVE OF Z WITH RESPECT TO S AND A
C      DERIVATIVE OF Q WITH RESPECT TO S AND A
C      DERIVATIVE OF P WITH RESPECT TO S AND A
       DO 436 J=1,I
       DZS(J)=((H(J)-A)**2-S*S)/S**3*Z(J)
       DZA(J)=(H(J)-A)/(S*S)*Z(J)
       DQS(J)=-(H(J)-A)/(Q(J)*S)**2*Z(J)
       DQA(J)=-Z(J)/(Q(J)**2*S)
       DPS(J)=(H(J)-A)/(P(J)*S)**2*Z(J)
       DPA(J)=Z(J)/(P(J)*P(J)*S)
   436 CONTINUE
       SUM1=0.0
       SUM2=0.0
       SUM3=0.0
       SUM4=0.0
       SUM5=0.0
       SUM6=0.0
       SUM=0.0
       SSUM=0.0
       DO 555 J=2,I
       SUM1=SUM1+XNF(J-1)*(DZS(J-1)/Q(J-1)+DQS(J-1)*Z(J-1))-XN(J)*(DZS(J)
      1/P(J)+DPS(J)*Z(J))
       SUM2=SUM2+XNF(J-1)*(Z(J-1)/Q(J-1))-XN(J)*(Z(J)/P(J))
       SUM3=SUM3+XNF(J-1)*(DZA(J-1)/Q(J-1)+DQA(J-1)*Z(J-1))-XN(J)*(DZA(J)
```

```
      1/P(J)+DPA(J)*Z(J))
       SUM4=SUM4+XNF(J-1)*((H(J-1)-A)*DZS(J-1)/Q(J-1)+(H(J-1)-A)*DQS(J-1)
      1*Z(J-1))-XN(J)*((H(J)-A)*DZS(J)/P(J)+(H(J)-A)*DPS(J)*Z(J))
       SUM5=SUM5+XNF(J-1)*((H(J-1)-A)*Z(J-1)/Q(J-1))-XN(J)*((H(J)-A)*Z(J)
      1/P(J))
       SUM6=SUM6+XNF(J-1)*(H(J-1)*DZA(J-1)/Q(J-1)+H(J-1)*DQA(J-1)*Z(J-1)-_
      1Z(J-1)/Q(J-1)-A/Q(J-1)*DZA(J-1)-A*Z(J-1)*DQA(J-1))-XN(J)*(H(J)/P(J
      2)*DZA(J)+DPA(J)*H(J)*Z(J)-Z(J)/P(J)-A*Z(J)*DPA(J)-A/P(J)*DZA(J))
       SUM=SUM+XNF(J-1)*(Z(J-1)/Q(J-1))-XN(J)*(Z(J)/P(J))
       SSUM=SSUM+XNF(J-1)*((H(J-1)-A)*Z(J-1)/Q(J-1))-XN(J)*((H(J)-A)*Z(J)
      1/P(J))
  555 CØNTINUE
      DERIVATIVE ØF F1(S,A) WITH RESPECT TØS AND A
      DERIVATIVE ØF F2(S,A) WITH RESPECT TØ S AND A
      FAS=F1(S,A) , SAS=F2(S,A)
      DFDS=1.0/S*SUM1-1.0/S**2 *SUM2
      DFDA=1.0/S*SUM3
      DSDS=1.0/S**2 *SUM4-2.0/S**3 *SUM5
      DSDA=1.0/S**2 *SUM6
      FAS=1.0/S*SUM
      SAS=1.0/S**2*SSUM
  303 D=DFDS*DSDA-DFDA*DSDS
      CØRRECTIØN FØR S
      XH=(DFDA*SAS-DSDA*FAS)/D
      CØRRECTIØN FØR A
      XK=(DSDS*FAS-DFDS*SAS)/D
      CØRRECTED A
      AA=A+XK
      CØRRECTED S
      SS=S+XH
   17 S=SS
      A=AA
      VALUE ØF F1(A,S) AND F2(A,S) ØBTAINED FRØM FINAL A AND S
      ZERØ1=0.0
      ZERØ2=0.0
      DØ 69 J=2,I
      ZER 1=ZERØ1+XNF(J-1)*(Z(J-1)/Q(J-1))-XN(J)*(Z(J)/P(J))
   69 ZERØ2=ZERØ2+XNF(J-1)*((H(J-1)-A)*Z(J-1)/Q(J-1))-XN(J)*((H(J)-A)*Z(
      1J)/P(J))
      IF(ABS(ZERØ1)-1.E-4) 68,68,15
   68 IF(ABS(ZERØ2)-1.E-5) 16,16,15
   15 ITCNT = ITCNT + 1
      IF(ITCNT - 100  ) 151,16,16
  151 GØ TØ 6
   16 WRITE (6       ,6666)ZERØ1,ZERØ2,A,AA,S,SS
 6666 FØRMAT(50X,8H7ERØ1 = ,E20.8//50X,8H7ERØ2 = ,E20.8//50X,9HNLMEAN =
      1,F10.8//50X,7HMEAN = ,F10.8//50X,15HNLSTND. DEV. = ,F10.8//50X,
      221HSTANDARD DEVIATIØN = ,F10.8//)
      WRITE (6       ,111)ITCNT
  111 FØRMAT(50X,20HNØ. ØF ITERATIØNS = ,I5//)
      CALL SØLVE(P,Q,XN,Y,Z,I,A,S,H,RELK,CØNK,CST,XNF )
      WRITE (6       ,77)
   77 FØRMAT(40X,49HCALCULATIØN ØF ALL-FIRE AND NØ-FIRE ENERGY LEVELS//)
      WRITE (6       ,971)GSQRD,HSQRD,SIGSQA,SIGSQS,SIGMAP,FIR50,   FNØT50
      1,RELV,CØNLV,FIREK,XRELV,CØNLV,FNØTK
  971 FØRMAT(50X 11HGSQUARED = ,F10.6//50X,10HI SQUARED =,F10.6//50X,29HS
```

```
1TAND. ERROR OF MEAN SQRD. = ,F10.^//50X,36HSTAND. ERROR OF STAND.
2DEV. SQRD. = ,F10.8//50X,35HSTAND. ERROR AT P PERCENT HEIGHT = ,F1
30.8//50X,33H99.9FIRE AT 50CONFIDENCE LEVEL = ,F10.8//50X,32H0.1FIR
4E AT 50CONFIDENCE LEVEL = ,F10.8//50X,F6.2,7HFIRE AT,F6.2,19HCONFI
5DENCE LEVEL = ,F10.8//50X,F6.2,7HFIRE AT,F6.2,19HCONFIDENCE LEVEL
6= ,F10.8// )
 GO TO 1
 END
```

```
      SUBRØUTINE PRØB(      H,X,Y,Z,A,S,P,C,I)
      DIMENSIØN              H(99),X(99),Y(99),Z(99),P(99),Q(99)
      DØ 7 J = 1, I
      HMASQ = (H(J)-A)**2
      EXPNT = HMASQ/(2.*S*S)
      Z(J) = (1./2.50663)/EXP(EXPNT)
      Y(J) = (H(J)-A)/S
      X(J) = ABS(Y(J))
      IF (X(J)-1.20 ) 21,21,22
   21 XSQ = X(J)*X(J)
      P(J)=0.79788455*X(J)*(0.99999774-XSQ*(0.16659433-XSQ*(.24638310E-0
   11-XSQ*.23974867E-02)))
      IF ( Y ( J ) ) 400, 401, 402
  400 P ( J ) = 0.5 - P ( J ) / 2.0
      GØ TØ 403
  401 P ( J ) = 0.5
      GØ TØ 403
  402 P(J)=P(J)/2.0+0.5
  403 Q ( J ) = 1.0- P ( J )
      GØ TØ 7
   22 IF (X(J) - 2.90 ) 25,26,26
   25 XSQ = X(J) * X(J)
      P(J) = 1.0
      PT = 1.0
      FACT = 1.0
      ØDD = 3.0
   29 PT = -PT*XSQ/(2.0*FACT )
      T = PT/ØDD
      P(J) = P(J) + T
      IF (ABS(T) - 0.00007 ) 63 ,70 ,70
   70 FACT = FACT + 1.0
      ØDD = ØDD + 2.0
      GØ TØ 29
   63 P(J) = 0.79788455*X(J)*P(J)
      IF ( Y ( J ) ) 500, 501, 502
  500 P ( J ) = 0.5 - P ( J ) / 2.0
      GØ TØ 503
  501 P ( J ) = 0.5
      GØ TØ 503
  502 P(J)=P(J)/2.0+0.5
  503 Q ( J ) = 1.0- P ( J )
      GØ TØ 7
   26 RXSQ = 1.0/(X(J)*X(J))
      P(J) = 1.0-0.79788455*EXP(-X(J)*X(J)/2.0)/X(J)*(1.0-RXSQ*(1.0-RXSQ
   1*   (3.0-RXSQ*(15.0-RXSQ*105.0))))
      IF ( Y ( J ) ) 600, 601, 602
  600 P ( J ) = 0.5 - P ( J ) / 2.0
      GØ TØ 603
  601 P ( J ) = 0.5
      GØ TØ 603
  602 P(J)=P(J)/2.0+0.5
  603 Q ( J ) = 1.0- P ( J )
    7 CØNTINUE
      RETURN
      END
```

```
      SUBROUTINE FØREST(FIX,FAX,N,X,U,V,IDENT )
      ALCG(W)=.4343*ALØG(W)
      ANT(W)=EXP(W/.4343)
      DIMENSIØNX(50),IDENT(20),FIX(99),FAX(99),S (30,15)
      DATA  S                    /64*0.,25.,40.,43.,46.,49.,51.,16*0.,2
     15.,27.,31..,35.,37.,42.,44.,47.,49.,51.,12*0.,25.,26.,28.,29.,31.,3
     24.,36.,39.,40.,43.,46.,48.,49.,51.,9*0.,25.,27.,28.,31.,32.,34.,36
     3.,37.,39.,40.,41.,42.,44.,46.,48.,49.,51.,4*0.,25.,26.,27.,27.,28.
     4,30.,30.,31.,32.,33.,35.,36.,37.,39.,41.,42.,43.,44.,46.,49.,49.,5
     51.,25.,26.,27.,27.,28.,29.,30.,30.,31.,32.,33.,34.,35.,36.,37.,38.
     6,40.,41.,42.,43.,45.,46.,47.,49.,51.,52.,25.,25.,26.,27.,29.,29.,3
     70.,30.,32.,32.,33.,34.,35.,36.,38.,39.,40.,42.,43.,44.,45.,46.,47.
     8,49.,51.,52.,25.,26.,27.,27.,29.,29.,30.,31.,32.,33.,34.,35.,36.,3
     97.,38.,39.,40.,41.,43.,44.,45.,47.,49.,50.,51.,53.,26.,26.,27.,28.
     A,29.,30.,31.,31.,32.,34.,35.,36.,37.,39.,39.,40.,41.,42.,44.,44.,4
     B5.,47.,49.,50.,52.,53.,26.,27.,28.,28.,29.,31.,31.,33.,34.,34.,36.
     C,37.,38.,39.,40.,41.,42.,43.,44.,45.,46.,47.,49.,51.,52.,53.,26.,2
     D7.,28.,29.,30.,31.,32.,33.,34.,35.,36.,37.,38.,39.,40.,41.,42.,43.
     E,44.,45.,46.,47.,49.,50.,52.,54./
      READ (5,639)(IDENT(J),J=1,20),N,C,D
  639 FØRMAT(20A1,I5,2F10.0)
      READ(5,637) (X(I),FIX(I),I=1,N)
  637 FØRMAT(6(F8.0,F4.0))
      READ(5,638) (FAX(I),I=1,N)
  638 FØRMAT( 6F10.0  )
      AFIX=0.0
      AFAX = 0.0
      BFIX=0.0
      BFAX=0.0
      SFIX=0.0
      SFAX=0.0
      I=0
 3010 W=I
      AFI. AFIX+W*FIX(I+1 )
      AFAX=AFAX+W*FAX(I+1)
      BFIX = BFIX + (W**2 ) * FIX(I+1)
      BFAX=BFAX+(W**2)*FAX(I+1)
      SFIX=SFIX+FIX(I+1)
      SFAX=SFAX+FAX(I+1)
      I=I+1
      IF(I-N+1)3010,3010,1115
 1115 R = D
      TFIX=((SFIX*BFIX)-AFIX**2)/(SFIX**2)
      TFAX=((SFAX*BFAX)-AFAX**2)/(SFAX**2)
   13 UFIX=C+R*(AFIX/SFIX-0.5)
      UFAX=C+R*(AFAX/SFAX+0.5)
      IF(TFIX-0.3) 118,23,23
  118 CIX=TFIX*100.
      LIX=CIX
 4111 WEEIX = 500.
      I=0
  112 WEEIX = AMIN1(WEEIX,ABS(UFIX-ALCG(X(I+1))))
      I=I+1
      IF(I-N+1)112,112,113
  113 RIX=ABS(WEEIX/R)
```

```
      SUBRØUTINE SØLVE (P,Q,XN,Y,Z,I,A,S,H,RELK,CØNK,CST,XNF )
      CØMMØN GSQRD,HSQRD,SIGSQA,SIGSQS,SIGMAP,FIR50, FNØT50,FIREK,FNØTK
      DIMENSIØN H(99),P(99),Q(99),XN(99),Y(99),Z(99),W(99),XNF(99)
      SUMNFJ=0.0
      SUMNJ = 0.0
      WJSUM = 0.0
      GSUM = 0.0
      HSUM = 0.0
      IZERØ = I/2 + 1
      IM1 = IZERØ - 1
      DØ 623 J = 1,I
      JM1 = J - 1
      W(J) = 1.0
      IF(H(J) - H(IZERØ )) 358,347,223
 358  DØ 6252 K = J,IM1
6262  W(J) = W(J) * P(K) / Q(K)
      GØ TØ 623
 347  W(J) = 1.0
      GØ TØ 623
 223  DØ  727 L = IZERØ , JM1
 727  W(J) = W(J) * Q(L) /_P(L)
 623  CØNTINUE
      DØ 741 J = 1,I
      WJSUM = WJSUM + W(J)
      SUMNFJ=SUMNFJ+XNF(J)
 741  SUMNJ = SUMNJ + XN(J)
      DØ 61 J = 2, I
      GSUM = GSUM + W(J-1)*(Z(J-1)**2/Q(J-1)**2+Z(J)**2/P(J)**2)
  61 _HSUM = HSUM+W(J-1)*(Y(J-1)**2*Z(J-1)**2/Q(J-1)**2+Y(J)**2*Z(J)**2/.
     1P(J)**2)
      GSQRD = WJSUM/GSUM
      HSQRD = WJSUM/HSUM
      SIGSQA=S*S/SUMNJ*GSQRD
      SIGSQS = S*S/SUMNJ*HSQRD
      SGSQAF=S*S/SUMNFJ*GSQRD
      SGSQSF=S*S/SUMNFJ*HSQRD
      SIGMAP = SQRT(SIGSQA+RELK**2*SIGSQS )
      SGMAPF=SQRT(SGSQAF+RELK**2*SGSQSF)
      FIR50 = A+RELK*S
      FNØT50 = A-RELK*S
      FIREK = FIR50+CØNK*((SUMNJ+CST)/SUMNJ)*SIGMAP
      FNØTK = FNØT50-CØNK*((SUMNJ+CST)/SUMNJ)*SIGMAP
      FIREKF=FIR50+CØNK*((SUMNFJ+CST)/SUMNFJ)*SGMAPF
      FNØTKF=FNØT50-CØNK*((SUMNFJ+CST)/SUMNFJ)*SGMAPF
      SIGSQA=SQRT((SIGSQA**2+SGSQAF**2)/2.)
      SIGSQS=SQRT((SIGSQS**2+SGSQSF**2)/2.)
      FIREK=SQRT((FIREK**2+FIREKF**2)/2.)
      FNØTK= SQRT(FNØTK * FNØTKF )
      RETURN
      END
```

48

```
      KIX = IFIX (RIX * 22. ) + 1
      VFIX=( S(LIX,KIX)/100.)*R
 2424 IF(TFAX-0.3)117,2525,2525
  117 CAX=TFAX*100.
      LAX=CAX
  114 WEEAX=500.
      I=0
  115 WEEAX = AMIN1(WEEAX,ABS(UFAX-ALCG(X(I+1))))
      I=I+1
      IF(I-N+1)115,115,116
  116 RAX=ABS(WEEAX/R)
      KAX = IFIX (RAX*22.)+1
      VFAX =.( S(LAX,KAX)/100.)*R
      GØ TØ 26
   23 VFIX=1.62*R*(TFIX+0.029)*SQRT(SFIX/(SFIX-1.)).
      GØ TØ 2424
 2525 VFAX=1.62*R*(TFAX+0.029)*SQRT(SFAX/(SFAX-1.))
   26 U=((SFAX*UFAX)+(SFIX*UFIX))/(SFAX+SFIX)
      V=SQRT((SFAX*(VFAX**2)+SFIX*(VFIX**2))/(SFAX+SFIX))
      RETURN
      END
```

M2 Squib Test D

CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.

ENERGY LEVEL	NO. OF FIRES/FAILS	
2.57287	0.	4.
2.59329	4.	10.
2.61278	10.	10.
2.63246	9.	2.
2.65321	1.	0.

AVG. = 2.60967

APPRX STND DEV = 0.02289

NO. OF ENERGY LEVELS = 5

ZER01 = 0.16199425E-04

ZER02 = 0.44016633E-06

NLMEAN = 2.61094216

MEAN = 2.61094216

NLSTND. DEV. = 0.02415103

STANDARD DEVIATION = 0.02415103

NO. OF ITERATIONS = 3

CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS

GSQUARED = 0.826383

HSQUARED = 1.496004

STAND. ERROR OF MEAN SQRD. = 0.00001932

STAND. ERROR OF STAND. DEV. SQRD. = 0.00003498

STAND. ERROR AT P PERCENT HEIGHT = 0.01916319

99.9FIRE AT 50CONFIDENCE LEVEL = 2.68556884

0.1FIRE AT 50CONFIDENCE LEVEL = 2.53631544

99.90FIRE AT 95.00CONFIDENCE LEVEL = 2.72416380

0.10FIRE AT 95.00CONFIDENCE LEVEL = 2.49772042

M6 Cap Test A

CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.

ENERGY LEVEL	NO. OF FIRES/FAILS	
0.14301	0.	2.
0.16435	3.	9.
0.18469	10.	10.
0.20683	11.	3.
0.22531	3.	0.

AVG. = 0.18533

APPRX STND DEV = 0.02337

NO. OF ENERGY LEVELS = 5

ZERO1 = 0.15720725E-05

ZERO2 = 0.10360964E-07

NLMEAN = 0.18430348

MEAN = 0.18430348

NLSTND. DEV. = 0.02588342

STANDARD DEVIATION = 0.02588342

NO. OF ITERATIONS = 4

CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS

GSQUARED = 0.835367

HSQUARED = 1.510502

STAND. ERROR OF MEAN SQRD. = 0.00002206

STAND. ERROR OF STAND. DEV. SQRD. = 0.00003989

STAND. ERROR AT P PERCENT HEIGHT = 0.01945744

99.9FIRE AT 50CONFIDENCE LEVEL = 0.26428325

0.1FIRE AT 50CONFIDENCE LEVEL = 0.10432371

99.90FIRE AT 95.00CONFIDENCE LEVEL = 0.305438

0.10FIRE AT 95.00CONFIDENCE LEVEL = 0.063157

CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.

ENERGY LEVEL	NO. OF FIRES/FAILS	
2.40993	0.	2.
2.42488	2.	9.
2.44091	9.	1?.
2.45484	12.	2.
2.46982	2.	0.

AVG. = 2.44130

APPRX STND DEV = 0.01477

NO. OF ENERGY LEVELS = 5

ZERO1 = 0.26669353E-04

ZERO2 = 0.75314892E-06

NLMEAN = 2.4413546?

MEAN = 2.44135469

NLSTND. DEV. = 0.01453041

STANDARD DEVIATION = 0.01453041

NO. OF ITERATIONS = 2

CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS

GSQUARED = 0.785916

HSQUARED = 1.171202

STAND. ERROR OF MEAN SQRD. = 0.00000663

STAND. ERROR OF STAND. DEV. SQRD. = 0.00000989

STAND. ERROR AT P PERCENT HEIGHT = 0.01005381

99.9FIRE AT 50CONFIDENCE LEVEL = 2.48625368

0.1FIRE AT 50CONFIDENCE LEVEL = 2.39645568

99.9)FIRE AT 95.00CONFIDENCE LEVEL = 2.50690502

0.10FIRE AT 95.00CONFIDENCE LEVEL = 2.37580431

APPENDIX C

TABLE

TABLE 1

EFFECT OF SAMPLE SIZE ON THE MAXIMUM LIKELIHOOD ESTIMATES OF THE MEAN AND STANDARD DEVIATION AND ON THE APPROXIMATIONS OF THESE ESTIMATES

Lot	Sample Size	Mean				Standard Deviation			
		Maximum Likelihood	Approximation	Absolute Difference	Relative Difference	Maximum Likelihood	Approximation	Absolute Difference	Relative Difference
1	20	2.18195006	2.17227	0.000968	0.44%	0.10385376	0.09964	0.00421	4.05%
2	20	2.21658185	2.19198	0.00246	1.11	0.16553961	0.13820	0.02734	16.52
3	20	2.13827476	2.13283	0.00054	0.25	0.05621809	0.04˙5	0.00846	15.05
4	20	2.20772761	2.19394	0.00138	0.63	0.12787458	0.11754	0.010335	8.08
5	20	2.19128922	2.16616	0.00251	1.15	0.16074003	0.14234	0.01840	11.45
6	20	2.08996966	2.08230	0.00077	0.37	0.10841461	0.10168	0.00674	6.22
1	30	2.19008294	2.18533	0.000475	0.22	0.08301159	0.07816	0.00485	5.84
2	30	2.21506256	2.19968	0.00154	0.68	0.12404235	0.10417	0.01987	16.02
3	30	2.12057662	2.11348	0.00071	0.33	0.07922158	0.06395	0.01527	19.27
4	30	2.22780049	2.21348	0.00143	0.64	0.10541917	0.09108	0.01434	13.60
5	30	2.20095909	2.18977	0.00112	0.51	0.18969359	0.14878	0.04091	21.57
6	30	2.23084623	2.21044	0.00204	0.91	0.199277325	0.17130	0.027977	14.04
1	50	2.16937867	2.16685	0.000253	0.12	0.11199255	0.10624	0.00575	5.13
2	50	2.18294474	2.17408	0.00089	0.41	0.09656488	0.08212	0.01445	14.96
3	50	2.16325137	2.15644	0.00680	3.14	0.12018208	0.10825	0.01193	9.93
4	50	2.22341090	2.20949	0.000139	0.06	0.09178600	0.08005	0.01174	12.79
5	50	2.17560852	2.16770	0.00044	0.20	0.12332264	0.11550	0.00782	6.34
6	50	2.17406946	2.16371	0.00104	0.48	0.16109429	0.14713	0.01396	8.67
7	50	1.85561229	1.85559	0.000002	0	0.03008042	0.03016	0.00008	0.27
8	50	2.60968155	2.60972	0.000004	0	0.03420447	0.03689	0.00269	7.86
9	50	2.48498195	2.48500	0.000002	0	0.01677419	0.01725	0.000476	2.84
10	50	2.60874438	2.60966	0.00012	0.05	0.02079704	0.02159	0.00081	3.89
11	50	2.65637481	2.65678	0.00104	0.39	0.03510309	0.03586	0.00076	2.17
12	50	2.59181881	2.59000	0.00182	0.70	0.02992462	0.030271	0.000346	1.16
13	50	2.64898539	2.64875	0.00002	0	0.02977113	0.02975	0.00002	0.07
14	50	2.63433656	2.63400	0.00034	0.13	0.02211387	0.02132	0.00079	3.57
1	100	2.19052792	2.18873	0.000180	0.08	0.11471657	0.10855	0.00617	5.38
2	100	2.20945472	2.20808	0.00014	0.06	0.09797628	0.09314	0.00484	4.94
3	100	2.17530292	2.16557	0.00097	0.45	0.12988110	0.12325	0.00663	5.10
4	100	2.21319243	2.20060	0.00126	0.57	0.09714309	0.09394	0.00320	3.29
5	100	2.22595269	2.22102	0.00044	0.20	0.15329798	0.14411	0.00919	5.99
6	100	2.18726733	2.17376	0.00135	0.62	0.13670944	0.13304	0.00367	2.68

Note: Values are in Log units. Relative Difference = $\dfrac{\text{Absolute Difference}}{\text{Maximum Likelihood Value}} \times 100$

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4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(Last name, first name, initial)*

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Statement A

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
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13 ABSTRACT

The maximum likelihood equations used to determine estimates of the mean and standard deviation for the Bruceton method of statistical analysis were solved numerically. A digital computer algorithm was developed to reduce the time associated with the numerical calculations. Test data was evaluated to determine the relative difference between values of the mean and standard deviation as obtained from the numerical solution of the maximum likelihood equations and as obtained from formulae approximating the maximum likelihood equations. The results showed that the differences between standard deviations were more pronounced.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Maximum Likelihood Equations						
Mean						
Standard Deviation						
Bruceton Statistical Analysis						
Newton-Raphson Method						
Probability of Explosion or Non-Explosion						

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